JOURNAL OF COMPUTATIONAL PHYSICS **48**, 150–156 (1982)

Note

Equivalence and Singularities: An Application of Computer Algebra

It is argued that, by computing a few scalar invariants, one may, in practical cases, obtain enough information about the equivalence problem, the Petrov type, and the physical interpretation of a metric through the characterization of its singularities. An interactive algorithm is developed and an application to the Harrison metrics described. In particular, one finds that, allowing for complex coordinate transformations, there are only two distinct Harrison type D metrics.

Algebra systems suitable for general relativity calculations are becoming widely available [1, 2]. Typically, given the metric tensor in a coordinate frame, these programs compute the Christoffel symbols, the components of the Riemann tensor, and their contractions. A few [2, 3] do the computations with respect to a moving frame.

For someone more interested in general relativity than in symbolic computing, the basic question is how to use these powerful tools to obtain sensible new information about the solutions of Einstein's equation. One much-studied [4–7] question is the computation of the Petrov type of a metric. One algorithm for this purpose will surely become a standard feature in any future relativity-oriented algebra system. Once a metric is Petrov classified, the next interesting questions are:

(1) Is this metric related to one already studied?

(2) What is the coordinate independent physical content of the metric?

(3) Is there a coordinate system where the physical content is transparent and further detailed study can be carried out?

We will be primarily concerned in this paper with questions (1) and (2). For a discussion of the problems involved when tackling question (3), see [8].

Question (1) carries the general name of *the equivalence problem.* From a theorem of Cartan [9], one knows that for each Riemannian manifold there is a number p such that another Riemannian manifold of the same dimension is locally equivalent to it iff there is a choice of tetrad basis that equals their Riemann tensor components together with those of the first p covariant derivatives. Then p is the smallest integer for which the covariant pth derivative can be written as a function of lower order covariant derivatives. The nature of the equivalence procedure following from this theorem has been discussed by Brans [10], and there is a recent computer implementation by Karlhede and Åman [11]. The procedure contains a nonalgorithmic step,

0021-9991/82/100150–07$02.00/0

where one must look for a solution to the algebraic relations obtained by equating the two independent sets of coordinate invariant tetrad components.

Actually, the essence of the equivalence problem is to find the change of coordinates $\bar{x}=f(x)$ that maps one metric into the other or to prove that no such function can exist. For this purpose it may not be necessary to obtain the complete invariant characterization of the geometry of both spaces as in the Cartan-inspired procedures. Realizing this simple fact and recalling past attempts to formulate the equivalence problem through the scalar invariants [12], we were led to propose a step-by-step procedure where

(a) At first a few scalar invariants of lowest order are computed for both metrics;

(b) One checks whether the algebraic relations obtained by equating the invariants are incompatible. If they are, the metrics are not equivalent and the procedure steps. Otherwise,

(c) One uses the algebraic relations to change coordinates as much as possible in one of the metrics. If the set of algebraic relations is not complete, one tries to guess the remaining coordinate changes needed to prove equivalence. If this fails one goes to (a) and computes some more invariants.

The steps (b) and (c) have the same nonalgorithmic nature as the last step in the Karlhede–Åman procedure. To our surprise, in the cases tested we have not had to compute invariants involving more than one covariant derivative of the Riemann tensor. Also, we found that in some cases the partial change of variables implied by an incomplete set of algebraic relations reduces the problem to an equivalence problem in lower dimensions.

In general relativity, two metrics are physically equivalent when there exists a *real* coordinate transformation between them. When two real metrics are related only by a *complex* coordinate transformation (i.e., when they are analytic continuations of each other), although nonequivalent, their main features are closely related. Therefore in the applications of our method we look also for complex coordinate transformations and denote this relation by $\mathbb{C}$-equivalence.

That the scalar invariants seem to be a handy tool for the equivalence problem is a fortunate circumstance because they are also the natural tool to classify spacetime singularities. Namely, a singularity is scalar if some polynomial in $R_{\mu\nu\alpha\beta}$ is ill behaved and C^n nonscalar if polynomials in $R_{\mu\nu\alpha\beta;\sigma_1,\ldots,\sigma_k}(k\leqslant n)$ are well behaved [13]. In this way, with the same computation one extracts information about the equivalence problem and the singularities; knowledge of the singularities is one the most important items in identifying the physical content of the metric (question 2). In this connection, it is important to be able to carry out the calculations until reaching, at least, polynomials in $R_{\mu\nu\alpha\beta;\sigma}$ because they are expected to blow up at an intermediate singularity [14].

Although the popular algorithms to compute the Petrov type of a metric use the Newman–Penrose spin coefficient formalism, Harris and Zund [7] have noticed that the classification of the metric may be obtained by checking certain algebraic relations between the scalar invariants. Therefore a Petrov type identifying capability may also be implemented in the same algorithm.

To close this introduction, one should mention that another computer algebra related problem that might be worthwhile to look at, and that has not so far attracted much attention [15], is

(4) Can one use computer algebra to generate new solutions to Einstein's equation?

Eventually, this may be better served by the new predicative logic based languages than by the traditional list or string processing symbolic languages.

Two algorithms to implement our program were written on top of a REDUCE [16] system implemented on a DEC-10. One of the algorithms is devised for general metrics and the other is specialized to diagonal metrics with the scalar invariants written out explicitly as program lines to save time by avoiding all contraction loops.

Both algorithms contain three blocks. The first one uses the standard routines to compute, given a metric, the Riemann, Ricci, and Weyl tensor. The second block converts the tensors to dyadic form and computes the scalar invariants of the first kind (those involving only the Riemann and metric tensors). Of the 14 scalar invariants of the first kind, we list the pure Weyl ones: (For vacuum metrics the remaining 10 vanish)

$$I^{(1)} = C^{\lambda\mu\nu k} C_{\lambda\mu\nu k}, \qquad I^{(3)} = \varepsilon^{\lambda\mu}_{\rho\sigma} C^{\rho\sigma\nu\alpha} C_{\lambda\mu\nu\alpha}/\sqrt{-g},$$

$$I^{(2)} = C_{\lambda\mu\nu\alpha} C^{\nu\alpha\rho\sigma} C^{\lambda\mu}_{\rho\sigma}, \qquad I^{(4)} = \varepsilon^{\lambda\mu}_{\tau\xi} C_{\lambda\mu\nu\alpha} C^{\nu\alpha\rho\sigma} C^{\tau\xi}_{\rho\sigma}/\sqrt{-g}.$$

This second block also checks the validity or nonvalidity of the relations

$$(I^{(1)}/48)^3 - 12(I^{(1)}/48)(I^{(3)}/192)^2 = (I^{(2)}/96)^2 - (I^{(4)}/192)^2,$$

$$6(I^{(1)}/48)^2 (I^{(3)}/192) - 8(I^{(3)}/192)^3 = 2(I^{(2)}/96)(I^{(4)}/192),$$

which hold for Petrov type D metrics. Finally the third block computes invariants that involve one covariant derivative of the Riemann tensor.

The polynomal factorization capabilities of REDUCE are very appropriate to obtain simple denominators in the invariants, thus providing information on singularities and change of variables even when the numerators are unmanageably long.

Interactive use of the programs is always advisable to decide when to introduce a simplification, whether an expression should or should not be expanded, etc.

We have tested our programs by carrying out a study of the Harrison metrics [17]. By transforming them to Kinnersley's form, d'Inverno and Russel-Clark [4] have reduced the 14 Harrison type D metrics to the following eight equivalence classes:

[III-1, III-4(a), III-4(b)]; [III-2, III-3, III-12(c)]; [III-8]; [III-7(a), III-7(b), III-7(c)]; [III-9(a)]; [III-9(b)]; [III-9(c)]; [III-10].

Åman and Karlhede [18] have recently found that the metrics III-9(a), III-9(b) and III-9(c) are equivalent, thus reducing the number of equivalence classes to six. Using our method and allowing for complex coordinate transformations, we found that there are only two distinct Harrison type D metrics, the $\mathbb{C}$-equivalent classes being:

[III-10, III-7(a), III-7(b), III-7(c), III-8, III-9(a), III-9(b), III-9(c)],
[III-1, III-2, III-3, III-4(a), III-4(b), III-12(c)]

We list in the Appendix the changes of coordinates that transform each metric into either III-10 or III-1. We denote by x^μ the coordinates of III-10 or III-1 and by $\bar{x}^\mu$ the coordinates of the metric to be transformed.

It should be mentioned that the trivial transformations taking III-7(a)), III-9(a), and III-8 into III-10 were used by Harrison himself [17] to generate two of these metrics.

Metric III-10 is the Schwarzschild metric, and III-1 a NUT metric (with $\mu_0 = \rho_0 = 0$). Hence all Harrison D metrics are equivalent to one of these or to an analytic continuation thereof. Equivalently, we might state that are all contained in the analytic continuations of the IV.B Kinnersley class (with $C = 0$ or $C = \frac{1}{2}$).

All this information was actually obtained by computing just the $I^{(1)}$ and $I^{(2)}$ invariants, with $I^{(3)}$ and $I^{(4)}$ being zero because the Harrison metrics are independent of one of the coordinates. For the two independent metrics we have

$$I^{(1)}_{\text{III-10}} = 192((x^3)^2 - l^2)^6/l^{16}, \qquad I^{(2)}_{\text{III-10}} = 768((x^3)^2 - l^2)^9/l^{24},$$
$$I^{(1)}_{\text{III}-1} = 12l^2/(x^0)^6, \qquad I^{(2)}_{\text{III}-1} = -12l^3/(x^0)^9.$$

To show that these metrics are inequivalent it sufficed to compute $K^{(1)} = R^{\alpha\beta\gamma\delta;\rho}R_{\alpha\beta\gamma\delta;\rho}$

$$K^{(1)}_{\text{III}-10} = (11520(x^3)^2/l^{24})((x^3)^2 - l^2)^8,$$
$$K^{(1)}_{\text{III}-1} = -180l^3/(x^0)^9.$$

To have both $I^{(i)}_{\text{III}-10} = I^{(i)}_{\text{III}-1}$ and $K^{(1)}_{\text{III}-10} = K^{(1)}_{\text{III}-1}$ would require $(x^3)^2 - l^2 = (x^3)^2$ which cannot hold if $l \neq 0$.

The scalar singularities of the metrics can be read from the invariants and correspond in III-10 to $x_3 \to \infty$ ($r \to 0$ in Schwarzschild coordinates) and $x^0 \to 0$ in III-1.

We have also computed the scalar invariants $I^{(1)}$ and $I^{(2)}$ for all Harrison type I vacuum metrics. (Following the tradition of not publishing results longer than one output page, we list below only two of the simplest ones, but will be glad to supply the others to anyone interested.)

$$I^{(1)}_{\text{III}-11}$$
$$= (3l^7/4(x^0)^6\,(x^1)^7 \sin^7(x^3/l)\,\lambda^4)$$
$$\times \{21(x^0)^2\,\lambda^4 - 24x^0(x^1)^{1/2}\,(\sin^{1/2}(x^3/l))\,l^{1/2}\lambda^2 + 16x^1 l \sin x^3/l\},$$

$$I^{(2)}_{\text{III}-11} = (3l^{21/2}/16(x^0)^9\,(x^1)^{21/2}\sin^{21/2}(x^3/l)\,\lambda^6) \times \{81(x^0)^3\lambda^6 - 180(x^0)^2\,(x^1)^{1/2}\sin^{1/2}(x^3/l)\,l^{1/2}\lambda^4 + 144x^0x^1\sin(x^3/l)\,l\lambda^2 - 64(x^1)^{3/2}\sin^{3/2}(x^3/l)\,l^{3/2}\},$$

$$I^{(1)}_{\text{I}-\text{B}-1(\text{a})} = (32l^4/F^2[(x^0)^2-(x^1)^2]^6)([x^0-x^1]/[x^0+x^1])^{2\alpha} \times \{(x^0)^4F^2(7F^2-3) - 9\alpha(x^0)^3x^1F^2(F^2-1) + 2(x^0)^2(x^1)^2(3F^4-10F^2+3) + 9\alpha x^0(x^1)^3(F^2-1) + (x^1)^4(7-3F^2)\},$$

$$I^{(2)}_{\text{I}-\text{B}-1(\text{a})} = (96l^6/\sinh^3(2x^3/l)[(x^0)^2-(x^1)^2]^6)([x^0-x^1]/[x^0+x^1])^{3\alpha} \times \{2(x^0)^6(5F^6-3F^4) + \alpha(x^0)^5x^1(-19F^4+18F^2-15) + 6(x^0)^4(x^1)^2(4F^6-7F^4+9F^2) + \alpha(x^0)^3(x^1)^3(-5F^6+39F^4-39F^2+5) + 6(x^0)^2(x^1)^4(-9F^4+7F^2-4) + \alpha x^0(x^1)^5(15F^4-18F^2+19) + 2(x^1)^6(3F^2-5)\},$$

where $F = \sinh(2x^3/l)$ and $\alpha = \pm\sqrt{2}$.

Preliminary work with these type I metrics led to the identification of the following $\mathbb{C}$-equivalence classes:

[II-A-4, II-A-5, II-B-2, II-C-3]; [II-A-2, II-B-1, II-C-2, II-A-3]; [II-A-6, II-C-4, II-B-3, II-A-7]; [III-11, III-12(a), III-12(b)]; [II-A-1, II-C-1]; [I-B-3, I-B-4]; [III-5]; [I-B-1(a)].

This grouping explored only the simplest invariant identifications and we believe a much greater reduction in the number of distinct types is possible.

Appendix

First Group (*$\mathbb{C}$-equivalent to* III-10)

III-7(a): $x^0 = i\bar{x}^2,\quad x^1 = i\bar{x}^1,\quad x^2 = i\bar{x}^0,\quad x^3 = \bar{x}^3,\quad l = \bar{l}.$

III-7(b): $x^0 = i\bar{x}^2,\quad x^1 = \bar{x}^1,\quad x^2 = -i\bar{x}^0 + \tfrac{1}{4}\pi\bar{l},\quad x^3 = \bar{x}^3,\quad l = \bar{l}.$

III-7(c): $x^0 = \bar{x}^2,\quad x^1 = \tfrac{1}{2}\bar{l}\cot^{-1} i\{(\bar{l}/4\bar{x}^1)\,e^{-2\bar{x}^0/\bar{l}} + ((-3\bar{l}/4\bar{x}^1) + (\bar{x}^1/l))\,e^{2\bar{x}^0/l}\},$
$x^2 = \tfrac{1}{2}\bar{l}\cosh^{-1}\{\tfrac{1}{2}e^{-2\bar{x}^0/\bar{l}} + \tfrac{1}{2}e^{2\bar{x}^0/\bar{l}} + (2(\bar{x}^1)^2/\bar{l}^2)\,e^{2\bar{x}^2/\bar{l}}\},\quad x^3 = \bar{x}^3,\quad l = \bar{l}.$

III-8: $x^0 = \bar{x}^2, \quad x^1 = \bar{x}^1, \quad x^2 = i\bar{x}^0, \quad x^3 = \bar{x}^3, \quad l = i\bar{l}.$

III-9(a): $x^0 = i\bar{x}^0, \quad x^1 = i\bar{x}^1, \quad x^2 = \bar{x}^2, \quad x^3 = \bar{x}^3, \quad l = -i\bar{l}.$

III-9(b): $x^0 = \bar{x}^0, \quad x^1 = \bar{x}^1, \quad x^2 = -\tfrac{1}{4}i\pi\bar{l} - \bar{x}^2, \quad x^3 = \bar{x}^3, \quad l = -i\bar{l}.$

III-9(c): $x^0 = i\bar{x}^0, \quad x^1 = \tfrac{1}{2}i\bar{l}\cot^{-1}\{(\bar{l}/4\bar{x}^1)\, e^{-2\bar{x}^2/\bar{l}} - ((3\bar{l}/4\bar{x}^1) + (\bar{x}^1/\bar{l}))\, e^{2\bar{x}^2/\bar{l}}\},$

$x^2 = \tfrac{1}{2}i\bar{l}\cosh^{-1}\{\tfrac{1}{2}e^{-2\bar{x}^2/\bar{l}} + (\tfrac{1}{2} + 2(\bar{x}^1)^2/\bar{l}^2)\, e^{2\bar{x}^2/\bar{l}}\}, \quad x^3 = \bar{x}^3, \quad l = i\bar{l}.$

Second Group ($\mathbb{C}$-equivalent to III-1)

III-2: $x^0 = -\lambda^{2/3}\bar{l}e^{\bar{x}^0 + \bar{x}^2}, \quad x^1 = i(e^{\bar{x}^0/2} + (\bar{l}^2/2\lambda^{4/3})\,\bar{x}^1 + \lambda^{2/3}\bar{l}^2 e^{(\bar{x}^0/2) + \bar{x}^2}),$

$x^2 = i\lambda^{1/3}\bar{l}\bar{x}^3, \quad x^3 = -e^{\bar{x}^0/2} + (\bar{l}^2/2\lambda^{4/3})\,\bar{x}^1 + \lambda^{2/3}\bar{l}^2 e^{(\bar{x}^0/2) + \bar{x}^2}, \quad l = \bar{l}.$

III-3: $x^0 = -4^{1/3}\bar{l}((\bar{x}^1/\bar{l}) \sin \bar{x}^3/l)^{1/2}, \quad x^1 = 4^{-1/3}i\bar{x}^0,$

$x^2 = 2^{1/3}i\bar{l}((\bar{x}^1/\bar{l}) \cos \bar{x}^3/l), \quad x^3 = 4^{-1/3}\bar{x}^2, \quad l = \bar{l}.$

III-4(a): $x^0 = -4^{1/3}l((\bar{x}^0/l) \sinh \bar{x}^3/l)^{1/2}, \quad x^1 = 4^{-1/3}\bar{x}^1,$

$x^2 = 2^{1/3}l((\bar{x}^0/l) \cosh \bar{x}^3/l), \quad x^3 = 4^{-1/3}\bar{x}^2, \quad l = \bar{l}.$

III-4(b): $x^0 = 4^{1/3}l((\bar{x}^0/l) \cosh \bar{x}^3/l)^{1/2}, \quad x^1 = 4^{-1/3}\bar{x}^1,$

$x^2 = 2^{1/3}l((\bar{x}^0/l) \sinh \bar{x}^3/l), \quad x^3 = 4^{-1/3}\bar{x}^2, \quad l = \bar{l}.$

III-12(c): $x^0 = -(k^{2/3}/\bar{l})\,\bar{x}^0\bar{x}^1 e^{\bar{x}^3/\bar{l}},$

$x^1 = i(\bar{x}^{0^{1/2}} e^{\bar{x}^3/2\bar{l}} + (\bar{l}^{1/2}/2k^{4/3})\,\bar{x}^0 e^{-\bar{x}^3/\bar{l}} + k^{2/3}\bar{x}^1 e^{\bar{x}^3/2\bar{l}}),$

$x^2 = ik^{1/3}\bar{x}^2,$

$x^3 = -\bar{x}^{0^{1/2}} e^{\bar{x}^3/2\bar{l}} + (\bar{l}^{1/2}/2k^{4/3})\,\bar{x}^0 e^{-\bar{x}^3/\bar{l}} + k^{2/3}\bar{x}^1 e^{\bar{x}^3/2\bar{l}}, \quad l = \bar{l}.$

Acknowledgments

The authors would like to express their thanks to the "Divisão de Informática" of the "Laboratório Nacional de Engenharia Civil" for a generous allotment of computer time.

References

1. For reviews and references see R. A. d'Inverno, *Gen. Rel. Grav.* **6** (1975), 567; H. I. Cohen, Ö. Leringe and Y. Sundblad, *Gen. Rel. Grav.* **7** (1976), 269; J. P. Fitch and H. I. Cohen, *Gen. Rel. Grav.* **11** (1979), 411.
2. I. Frick, Univ. of Stockholm Inst. of Phys. Report 77–14, 1977.
3. S. J. Campbell and J. Wainwright, *Gen. Rel. Grav.* **8** (1977), 987.
4. R. A. d'Inverno and R. A. Russel-Clark, *J. Math. Phys.* **12** (1971), 1258.
5. J. P. Fitch, in "Proceedings, 4th Symposium on Advanced Computational Methods in Physics," 1977.

6. J. WAINWRIGHT, *J. Math. Phys.* **18** (1977), 672.
7. R. A. HARRIS AND J. D. ZUND, *Commun. ACM* **21** (1978), 715.
8. R. A. D'INVERNO AND J. SMALLWOOD, *Gen. Rel. Grav.* **9** (1978), 195.
9. E. CARTAN, "Leçons sur la geometrie des espaces de Riemann," Gauthier–Villars, Paris, 1946.
10. C. H. BRANS, *J. Math. Phys.* **6** (1965), 94.
11. A. KARLHEDE AND J. E. ÅMAN, *in* "Proceedings, EUROSAM 79," Lecture Notes in Computer Science, Vol. 72, p. 42, Springer-Verlag, Berlin/New York, 1979.
12. W. KUNDT, *in* "Recent Developments in General Relativity," p. 307, Pergamon, Elmsford, N. Y., 1962.
13. G. F. R. ELLIS AND B. G. SCHMIDT, *Gen. Rel. Grav.* **8** (1977), 915.
14. S. T. C. SIKLOS, *Gen. Rel. Grav.* **10** (1979), 1003.
15. See, however, D. BARTON AND J. P. FITCH; *Commun. ACM*, **14** (1971), 542.
16. A. C. HEARN, "Reduce user's manual," 2nd ed., Univ. of Utah, 1973.
17. B. K. HARRISON, *Phys. Rev.* **116** (1959), 1285.
18. J. E. ÅMAN AND A. KARLHEDE, *Phys. Lett.* **80A** (1980), 229.

RECEIVED: July 20, 1981; REVISED: November 13, 1981

S. M. ELEUTÉRIO AND R. V. MENDES

CFMC–Instituto Nacional de Investigação Científica
1699 Lisboa, Portugal